Derived Categories

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October 15, 2023

1 Introduction

In this short note, I give a brief introduction to derived categories, aiming to define and describe basic properties of $D^b(X)$, the (bounded) derived category of coherent sheaves on a scheme X. These notes were written for the first two lectures in a learning seminar about Bridgeland stability, following [MS17]. The primary reference for this note is [Huy06]; other references include (but *definitely* are not limited to) [Bay11], [Tho00], and Akhil Mathew's blog post. These notes are meant to collect existing knowledge from the above sources and compile them into a shorter reference; as such, they will often be heavily based on the above references. For outside readers, this note could be useful if you have learned about chain complexes but want to know why we should use them, or if you simply want to learn about the derived category.

2 Motivation

Why should we care about derived categories?

We use chain complexes all the time: in algebraic topology, chain complexes compute homology and cohomology of topological spaces, giving us a variety of different invariants with which we can distinguish many spaces. In algebraic geometry, chain complexes also give us sheaf cohomology (among other cohomology theories), and coherent sheaves can arise as the (co)kernels of maps in complexes. In many cases, (co)homology groups are sufficient for whatever purposes we had. However, as any student in algebraic topology quickly learns, two spaces which are not homotopic may nevertheless have the same homology groups! The idea here is to view the **chain complexes** as the invariant, rather than the homology groups. In many cases, the chain complexes are able to differentiate the spaces, despite the homology groups being identical.

Example 2.0.1. There are pretty extreme examples where common invariants fail to distinguish topological spaces. For example, this StackExchange post describes two compact simply-connected manifolds with identical cohomology rings, homology groups, *and* homotopy groups, yet are not homotopy equivalent.

Example 2.0.2. For an algebraic example, the two complexes

$$\mathbb{C}[x,y]^{\oplus 2} \xrightarrow{(x,y)} \mathbb{C}[x,y] \quad \text{and} \quad \mathbb{C}[x,y] \xrightarrow{0} \mathbb{C}$$

have the same homology but are not quasi-isomorphic.

Another interesting place where chain complexes are used is in the definition of cohomology, constructed from homology. The goal is to identify cohomology as the dual of homology, i.e. $H^* = \mathrm{Hom}(H_*, \mathbb{Z})$. But when the homology groups carry torsion, cohomology cannot be defined so naively: in order to make cohomology a true dual to homology, we should be able to recover H_* from H^*, but this doesn't work because $\mathrm{Hom}(-, \mathbb{Z})$ is always torsion-free. In the standard definition of H^*, we are actually supposed to apply $\mathrm{Hom}(-, \mathbb{Z})$ at the level of chain complexes, and this indeed gives us a true dual: applying $\mathrm{Hom}(-, \mathbb{Z})$ twice to the chain complex does recover the original complex.

In summary, we see that we should **work with chain complexes rather than (co)homology groups**, and that's precisely what we aim to do with derived categories! To get right to it: **the derived category of a category $\mathcal{A}$ is essentially just all of the chain complexes in $\mathcal{A}$.** We'll make this more precise throughout this note!

3 Abelian and triangulated categories

Our ultimate goal is to work entirely with chain complexes formed out of some category $\mathcal{A}$. In order to do so, the category $\mathcal{A}$ needs to have several useful properties - for example, having kernels and cokernels. Our setting is that we assume $\mathcal{A}$ is an abelian category, and our goal will eventually be to construct and describe the derived category $D(\mathcal{A})$, which will turn out to be a triangulated category. In this section we'll give a quick introduction to abelian and triangulated categories. As a disclaimer, I'll assume some knowledge of homological algebra, which is discussed in great detail in [Huy06]; another great source is [Wei94].

3.1 Abelian categories

Definition 3.1.1. A category $\mathcal{A}$ is **additive** if for all objects A, B, the Hom-set $\mathrm{Hom}(A, B)$ is an abelian group such that:

(1) The composition maps $\mathrm{Hom}(A_1, A_2) \times \mathrm{Hom}(A_2, A_3) \to \mathrm{Hom}(A_1, A_3)$ are bilinear.
(2) There exists a zero object $0 \in \mathcal{A}$, such that $\mathrm{Hom}(0, 0)$ is the trivial group.

(3) For any two objects A_1, A_2, there is an object B which is the direct sum (and direct product) of these two objects.

Informally, an additive category is just a category where the Hom-sets are abelian groups, and thus carry an additive structure. Naturally, functors between additive categories are generally assumed to be additive as well, in the sense that the induced maps on Hom-groups is a group homomorphism. There is a very similar notion of **k-linear categories** for a field k, which is defined in pretty much the same way except that the Hom-sets are k-vector spaces and functors should be k-linear maps.

Definition 3.1.2. A category $\mathcal{A}$ is **abelian** if it is additive and additionally satisfies another condition:

(4) Every morphism f admits a kernel and a cokernel, and the natural map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

Why are abelian categories important? Well, they carry most of the fundamental properties we'd like in order to do homological algebra. In particular, exact sequences (usually) only make sense in abelian categories. The canonical example of an abelian category is AbGrp, the category of abelian groups. I often think of abelian categories as isolating the most important features of AbGrp and demanding they hold in whatever category we're actually working in.

Example 3.1.3. Here are some important examples of abelian categories.

(1) For a commutative ring R, the category $R-\mathsf{mod}$ of R-modules is abelian. Furthermore, the full subcategory of finitely generated modules is also abelian. In fact, the Freyd-Mitchell embedding theorem states that any (small) abelian category is a full subcategory of $R-\mathsf{mod}$ for some unital ring R (not necessarily commutative). (Unfortunately, projective and injective objects do not necessarily correspond to projective and injective R-modules under this correspondence.)

(2) Let X be a topological space. Then the category of sheaves of abelian groups, $\mathsf{Sh}(X)$, is abelian. More generally, for $\mathcal{O}$ a sheaf of commutative rings on X, the subcategory of $\mathcal{O}-\mathsf{mod}$ is abelian as well.

(3)* Let X be a scheme. Then the categories of **quasicoherent and coherent sheaves on X, $\mathsf{QCoh}(X)$ and $\mathsf{Coh}(X)$, are both abelian.** This example will be our main focus for the rest of the seminar!

3.2 Triangulated categories

In algebraic topology, we see long exact sequences arising from short exact sequences all the time. For example, let X be a topological space, and A, B be subspaces whose interiors cover X. Then Mayer-Vietoris gives us the long exact sequence of cohomology groups

$$\cdots \to H^{n-1}(A \cap B) \to H^n(X) \to H^n(A) \oplus H^n(B) \to H^n(A \cap B) \to H^{n+1}(X) \to \dots .$$

In some sense, we have a "short exact sequence" from the cohomology complex of X, to the direct sum of the cohomology complexes of A and B, to the cohomology complex of $A \cap B$... but not exactly, because it returns back to the cohomology complex of X, except shifted! This behavior of complexes is captured by the notion of an **exact triangle**, which are the analogues of short exact sequences in abelian categories. This leads to the notion of a triangulated category.

Definition 3.2.1. Let $\mathcal{D}$ be an additive category. The structure of a **triangulated category** on $\mathcal{D}$ is given by the data of:

- an additive equivalent $\Sigma : \mathcal{D} \to \mathcal{D}$, known as the **shift functor** (sometimes called translation functor), and
- a set of **exact triangles** (sometimes called distinguished triangles) $A \to B \to C \to \Sigma(A)$, subject to axioms TR1-TR4, discussed below.

Notation 3.2.2. For any $n \in \mathbb{Z}$ and any object $A \in \mathcal{D}$, we denote $A[n] := \Sigma^n(A)$. Similarly, for a map $f : A \to B$, denote by $f[n] := \Sigma^n(f)$ to be the corresponding map $A[n] \to B[n]$.

In the case of chain complexes, the shift functor will quite literally be shifting the indices, hence the name!

Let's go into detail about **triangles**, and especially the class of exact triangles. A **triangle** is denote by $A \to B \to C \to A[1]$. First, a morphism between two triangles is just a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1], \end{array}$$

and an isomorphism of triangles is a diagram such that f, g, h are all isomorphisms.

exact triangles are a special class of triangles which satisfy certain axioms in order to give them the properties reminiscent of short exact sequences. Let's discuss them.

TR1

(i) Any triangle of the form $A \xrightarrow{\text{id}} A \to 0 \to A[1]$ is exact.

(ii) The class of exact triangles is closed under isomorphism (of triangles).

(iii) Any morphism $f : A \to B$ can be completed to a exact triangle $A \xrightarrow{f} B \to C \to A[1]$.

Part (iii) is particularly important! It roughly says that "there are enough exact triangles." It looks a bit strange due to the non-uniqueness of the completing object, but actually it turns out to be unique, up to isomorphism! We call this the **cone** (of the morphism f), denoted by $\text{cone}(f)$. We will see an explicit construction of it in the derived category (working with chain complexes).

TR2

Exact triangles are invariant under "rotation." Concretely, $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is exact iff $B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$ is exact.

TR3

A morphism of exact triangles is given by three maps. However, any two maps (which form a commutative square on their respective objects) can be completed to a morphism of exact triangles.

In other words, suppose that there exists a commutative diagram of exact triangles with maps f, g:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1]. \end{array}$$

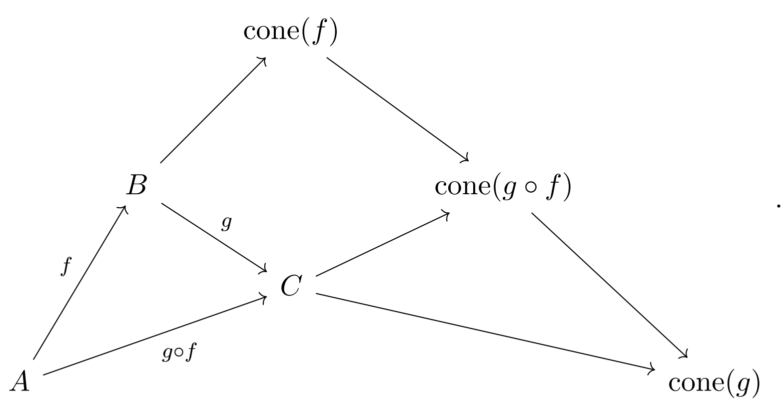
Then there exists a (not necessarily unique!) map $h : C \to C'$ which makes the diagram commute, i.e., completes f and g to a morphism between the two exact triangles.

TR4

This axiom is known as the octahedral axiom. Suppose we have a composition (**not** a exact triangle) $A \xrightarrow{f} B \xrightarrow{g} C$. The octahedral axiom states that **the cones of the morphisms $f, g, g \circ f$ form an exact triangle**. In other words, there is an exact triangle

$$\mathrm{cone}(f) \to \mathrm{cone}(g \circ f) \to \mathrm{cone}(g) \to \mathrm{cone}(f)[1],$$

captured succinctly in the following diagram (where all of the "lines" are exact triangles):



Exact triangles behave very similarly to short exact sequences. One reason why they're useful is that secretly, **long exact sequences come from exact triangles**. One very important example is the following:

Proposition 3.2.3. *Let $A \to B \to C \to A[1]$ be an exact triangle. Then for any object A_0, we have long exact sequences*

$$\cdots \to \mathrm{Hom}(A_0, A[n]) \to \mathrm{Hom}(A_0, B[n]) \to \mathrm{Hom}(A_0, C[n]) \to \mathrm{Hom}(A_0, A[n+1]) \to \cdots,$$
$$\cdots \to \mathrm{Hom}(C[n], A_0) \to \mathrm{Hom}(B[n], A_0) \to \mathrm{Hom}(A[n], A_0) \to \mathrm{Hom}(C[n-1], A_0) \to \cdots.$$

Typically, when we deal with functors between triangulated categories, we'd like for them to be **exact**, which means that they send exact triangles to exact triangles and they commute with the shift functor.

For more on the properties of triangulated categories, see [Huy06].

4 Derived categories

In this section, we describe our primary object of interest: the derived category. Our starting point is, of course, the category of complexes.

4.1 Kom($\mathcal{A}$) and $D(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category.

Definition 4.1.1. The **category of complexes** $\mathrm{Kom}(\mathcal{A})$ is the category whose objects are complexes $A^\bullet \in \mathcal{A}$ and whose morphisms are morphisms of complexes.

It is fairly easy to see that **Kom($\mathcal{A}$) is an abelian category**.

Note that we may identify $\mathcal{A}$ as a full subcategory of $\mathrm{Kom}(\mathcal{A})$ by sending $A \in \mathcal{A}$ to the complex concentrated in degree 0 (whose 0th term is A).

There is no problem defining a shift functor Σ: we simply "shift" the complex. More precisely, suppose $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ with differentials $d_A^\bullet$. Then $(A^\bullet[1])^i := A^{i+1}$ and differential $d^i_{A[1]} := -d_A^{i+1}$ (note the difference in the differential!). This shift functor defines an equivalence of abelian categories $\mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$. Furthermore, we again have the notion of cohomology functors $H^i : \mathrm{Kom}(\mathcal{A}) \to \mathcal{A}$, taking a complex and returning the ith cohomology "group."

Is the story over now? Unfortunately, no. The category $\mathrm{Kom}(\mathcal{A})$ is lacking in some important departments. One is that $\mathrm{Kom}(\mathcal{A})$ is not a triangulated category, at least not with canonical choices for exact triangles. For example, short exact sequences of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ (with the zero map for $C^\bullet \to A^\bullet[1]$) **do not** satisfy the requirements to be an exact triangle. Another is due to one of our original reason for working with complexes: we wish for complexes to tell us when things are isomorphic. The corresponding notion of "isomorphism" for complexes is **quasi-isomorphism** (**qis** for short), which is a map of complexes which induces an isomorphism on every (co)homology group. Unfortunately, one major issue with complexes is that even when the original objects were isomorphic, the corresponding complexes may only have a quasi-isomorphism in one direction! This doesn't seem to be the answer, then: we want the complexes to truly be isomorphic in the category. This is the main idea behind the derived category: we essentially "invert" the quasi-isomorphisms so that they are true isomorphisms.

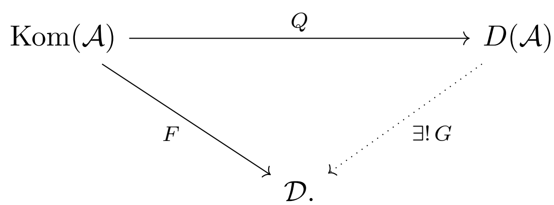
Theorem 4.1.2. *There exists a category $D(\mathcal{A})$, called the **derived category of $\mathcal{A}$**, along with a functor*

$$Q : \mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$$

such that:

(1) If $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism, then $Q(f)$ is an isomorphism in $D(\mathcal{A})$.

*(2) Q is **universal** with respect to this property, i.e. any functor $F : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ satisfying property (1) factors uniquely over Q, so that there is a unique functor (up to isomorphism) $G : D(\mathcal{A}) \to \mathcal{D}$ with $F \simeq G \circ Q$:*



This is just an existence theorem, but we'd like to get to know this category concretely. To do so, we actually need to pass through the homotopy category $K(\mathcal{A})$ first, but in my opinion this over-complicates what's going on. Therefore, I'll give two descriptions: one concrete and skipping all of the technical details, namely §4.2, and one slightly longer (but still skipping most of technical details, but at least covering an overview of the route), namely §4.3.

4.2 The short route

Let's get straight to the point.

Definition 4.2.1. The **derived category** $D(\mathcal{A})$ (namely the unique category satisfying Theorem 4.1.2) is the category obtained by taking the class of quasi-isomorphisms in $\mathrm{Kom}(\mathcal{A})$ and inverting all of them formally.

4.3 The slightly longer route

As mentioned before, in order to be a bit more precise, we need to pass through the homotopy category.

Definition 4.3.1. The **homotopy category of complexes** $K(\mathcal{A})$ is the category whose objects are $\mathrm{Obj}(\mathrm{Kom}(\mathcal{A}))$, and whose morphisms are just the morphisms of complexes up to chain homotopy, i.e. $\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) \coloneqq \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$.

Why should we pass to the homotopy category? There are several higher-level reasons: when quasi-isomorphisms are inverted, chain homotopies are in fact identified (akin to how when we localize a commutative ring R, then a previously nonzero element r may now be identified with $r/1 = 0 \in S^{-1}R$ after localization); another is that $K(\mathcal{A})$ is a triangulated category, and since the quasi-isomorphisms are form a localizing class in $K(\mathcal{A})$ (but **not** in $\mathrm{Kom}(\mathcal{A})$!), inverting them is very concrete and well-behaved, and therefore easy to write down! This is a more general procedure, known as localization of a category.

As a result $D(\mathcal{A})$ will be defined to be the category obtained by inverting (the classes of) quasi-isomorphisms in $K(\mathcal{A})$.

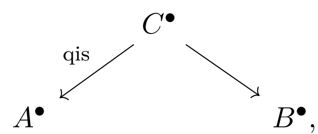
This may sound pretty stupid, given that we directly inverted in §4.2, but more technical details of the construction in the next section may shed some light on why this longer path is useful to think about.

4.4 Concrete description

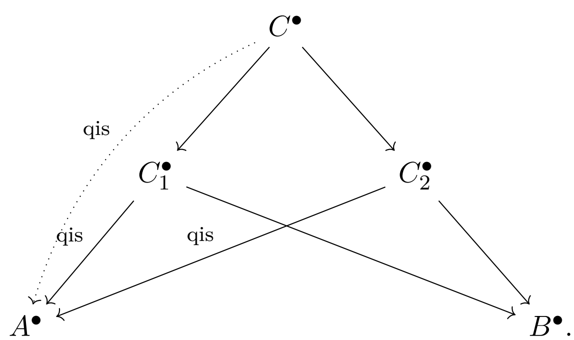
Ok, let's get to a concrete description of the derived category. First note that $D(\mathcal{A})$ **is a triangulated category**. Note: if you skipped §4.3, ignore every time $K(\mathcal{A})$ is mentioned, or blackbox it as an intermediate step.

The **objects** are just complexes in $\mathcal{A}$, i.e. the objects of $D(\mathcal{A})$ are precisely the objects of $\mathrm{Kom}(\mathcal{A})$.

The **morphisms** are more complicated. Let $A^\bullet$ and $B^\bullet$ be complexes. Then $\mathrm{Hom}_{D(\mathcal{A})}$ is the set of all equivalence classes of diagrams



where $C^\bullet \to A^\bullet$ is a quasi-isomorphism. (These are sometimes known as "roofs.") Two such diagrams *equivalent* if they are dominated by a third such diagram in the homotopy category; in other words, there should be a commutative diagram in $K(\mathcal{A})$ of the form



The reason that commutativity of this diagram is only required *up to homotopy* is because the construction of the cone, which will be done shortly, is unique only up to homotopy. (Another reason why passing to $K(\mathcal{A})$ makes sense!) It will turn out that

How does composition of two morphisms work? Given diagrams in $\mathrm{Hom}(A^\bullet, B^\bullet)$ and $\mathrm{Hom}(B^\bullet, C^\bullet)$ of the form

the composition of these two is defined to be a diagram (commutative in $K(\mathcal{A})$) of the form

$$\begin{array}{ccccccccc} & & & & C_0^\bullet & & & & \\ & & & \overset{\text{qis}}{\swarrow} & & \searrow & & & \\ & & C_1^\bullet & & & & C_2^\bullet & & \\ & \overset{\text{qis}}{\swarrow} & & \searrow & & \overset{\text{qis}}{\swarrow} & & \searrow & \\ A^\bullet & & & & B^\bullet & & & & C^\bullet. \end{array}$$

It turns out that such a diagram indeed always exists, and furthermore it is unique up to equivalence.

Example 4.4.1. In algebraic topology, the Whitehead theorem states that the underlying topological spaces $|X|$ and $|Y|$ of simplicial complexes X and Y are homotopy equivalent iff there are maps of simplicial complexes

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X & & & & Y \end{array}$$

inducing maps on homology $H_*(X) \xleftarrow{\sim} H_*(Z) \xrightarrow{\sim} H_*(Y)$. What it's *really* saying is that $|X|$ and $|Y|$ are homotopy equivalent iff there is a Z inducing quasi-isomorphisms

$$\begin{array}{ccccc} & & C_\bullet^Z & & \\ & \overset{\text{qis}}{\swarrow} & & \overset{\text{qis}}{\searrow} & \\ C_\bullet^X & & & & C_\bullet^Y. \end{array}$$

So this provides us with some motivation for defining roofs!

Example 4.4.2. One of the main problems with complexes is that quasi-isomorphic complexes may only have maps in one direction. Consider the following quasi-isomorphism of complexes in AbGrp:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}.$$

This is clearly a quasi-isomorphism of complexes, but **there is no quasi-isomorphism from the bottom complex to the top complex**, since there are no nontrivial maps from torsion to free. How do roofs solve this problem? Well, if $f : A^\bullet \xrightarrow{\text{qis}} B^\bullet$ is a quasi-isomorphism, we can construct a roof representing this map f:

$$\begin{array}{ccccc} & & A^\bullet & & \\ & \overset{\text{id}}{\swarrow} & & \overset{f}{\searrow} & \\ A^\bullet & & & & B^\bullet. \end{array}$$

Now it becomes clear how roofs solve this problem. To get a "map" in the other direction, we just switch the roles of $A^\bullet$ and $B^\bullet$, since both $\mathrm{id}_{A^\bullet}$ and f are quasi-isomorphisms:

$$\begin{array}{ccc} & A^\bullet & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle \mathrm{id}} \\ B^\bullet & & A^\bullet. \end{array}$$

We still have several things remaining to define from our knowledge of triangulated categories: shift functor, cones, and exact triangles.

The **shift functor** works exactly as it does in $\mathrm{Kom}(\mathcal{A})$: it shifts the indices of the complex.

Definition 4.4.3. Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes, living either in $K(\mathcal{A})$ or $D(\mathcal{A})$. We define its **mapping cone** (or simply **cone**) $\mathrm{cone}(f)$ to be the complex satisfying

$$\mathrm{cone}(f)^i = A^{i+1} \oplus B^i, \quad d^i_{\mathrm{cone}(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}.$$

The cone comes with natural morphisms of complexes $B^\bullet \to \mathrm{cone}(f)$ and $\mathrm{cone}(f) \to A^\bullet[1]$, namely the obvious ones: the injection $B^\bullet \hookrightarrow A^\bullet[1] \oplus B^\bullet$ and the projection $A^\bullet[1] \oplus B^\bullet \to A^\bullet$.

Remark 4.4.4. The inspiration for mapping cone comes from the topological mapping cone. It turns out that the complex of singular chains of the topological mapping cone for some map of topological spaces $f : X \to Y$ is homotopic to the algebraic mapping cone in Definition 4.4.3 of the induced map of complexes from the complex of singular chains of X to the complex of singular chains of Y. Thanks to Rosie Shen for pointing this out!

Example 4.4.5. Recall from Axiom TR1 that $A \xrightarrow{\mathrm{id}} A \to 0 \to A[1]$ is an exact triangle. But we defined the cone (Definition 4.4.3) some very large complex: namely, in this case, $\mathrm{cone}(\mathrm{id}_{A^\bullet}) = A^\bullet[1] \oplus A^\bullet$. This appears at first glance to be very different from 0. Now it *is* different from zero in that they are obviously different complexes, but that only means they are different in $\mathrm{Kom}(\mathcal{A})$. We have an obvious map of complexes $\mathrm{cone}(\mathrm{id}_{A^\bullet}) \to 0$. Directly from the directions of the differential of the cone, we see that in fact as it's defined, all of the cohomology groups of $\mathrm{cone}(\mathrm{id}_{A^\bullet})$ are zero! This means that the induced maps on cohomology $H^i(\mathrm{cone}(\mathrm{id}_{A^\bullet}) \to H^i(0)$ are just the maps $0 \to 0$, which are all isomorphisms, and hence the map $\mathrm{cone}(\mathrm{id}_{A^\bullet}) \to 0$ is a quasi-isomorphism, which means that in $D(\mathcal{A})$ these two objects are isomorphic. So we see that there is no issue after all!

Lastly, the **exact triangles** (in either $K(\mathcal{A})$ or $D(\mathcal{A})$) are simply defined to be the triangles which are isomorphic to some triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \to \mathrm{cone}(f) \to A^\bullet[1],$$

where the two unnamed maps are the natural maps from Definition 4.4.3.

The crucial point is that **these constructions and definitions give $K(\mathcal{A})$ and $D(\mathcal{A})$ the structure of a triangulated category.** Furthermore, the natural functor $Q_A : K(\mathcal{A}) \to D(\mathcal{A})$ is exact (as a functor of triangulated categories, which admittedly I didn't define, but essentially just takes exact triangles to exact triangles). Yay!

One final notational point: these complexes can be infinite in either (or both) directions. In practice, we often only work with bounded sequences – either bounded on both sides, or bounded on one side.

Definition 4.4.6. Let $\mathrm{Kom}^*(\mathcal{A})$ for $* = +, -, b$ be the category of complexes $A^\bullet$ with $A^i = 0$ for $i \ll 0$, $i \gg 0$. amd $|i| \gg 0$, respectively. By following the same procedure as above, we get $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ (note that the objects are the same as in $\mathrm{Kom}^*(\mathcal{A})$, so it is still very much concrete!).

Let's wrap up this subsection with some basic facts about the derived category.

Proposition 4.4.7. *(i) The cohomology objects $H^i(A^\bullet)$ for $A^\bullet \in D(\mathcal{A})$ are well-defined objects of $\mathcal{A}$ (in other words, taking the equivalence class "up to homotopy and inverting quasi-isomorphisms" doesn't change the cohomology of a complex).*

(ii) We can identify $\mathcal{A}$ as the full subcategory of $D(\mathcal{A})$ with cohomology objects $H^i(-) = 0$ for all $I \neq 0$, i.e. cohomology concentrated in degree 0. One direction is given by identifying an object of $\mathcal{A}$ with a complex concentrated in degree 0. The other direction is taking H^0 of a complex.

(iii) Let $0 \to A \xrightarrow{f} B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Then under the embedding $\mathcal{A} \hookrightarrow D(\mathcal{A})$ (or $K(\mathcal{A})$) this becomes an exact triangle $A \to B \to C \to A[1]$, where the last map is given by composing the quasi-isomorphism $C \to cone(f)$ and the natural map $cone(f) \to A[1]$. In other words, short exact sequences indeed give rise to exact triangles!

(iv) An exact triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ naturally induces a long exact sequence $\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$.

(v) The forgetful functors $D^*(\mathcal{A}) \to D(A)$ define equivalences with the full triangulated subcategories of all complexes $A^\bullet \in D(\mathcal{A})$ with $H^i(A^\bullet) = 0$ for $i \ll 0$, $i \gg 0$, and $|i| \gg 0$.

4.5 $D^b(X)$

The most important example is the **bounded derived category of coherent sheaves.**

Definition 4.5.1. Let X be a scheme. Then we define

$$D^b(X) \coloneqq D^b(\mathsf{Coh}(X))$$

to be the **bounded derived category of the (abelian) category of coherent sheaves on X.**

There is one minor problem: the category $\mathsf{Coh}(X)$ usually contains no non-trivial injective objects, which makes computations in $D^b(\mathsf{Coh}(X))$ as currently defined a pain. Fortunately, its relative $\mathsf{QCoh}(X)$ *does* contain enough injectives (and so does the category of $\mathcal{O}_X$-modules), **at least when X is noetherian**. As a result, $D^b(\mathsf{Coh}(X))$ finds itself naturally as a full triangulated subcategory of $D^b(\mathsf{QCoh}(X))$. As a result, **whenever we talk about $D^b(X)$, we assume X is noetherian.**

Proposition 4.5.2. *Let $D^b_{\mathsf{Coh}}(\mathsf{QCoh}(X))$ denote the full triangulated subcategory of $D^b(\mathsf{QCoh}(X))$ consisting of (bounded) complexes, all of whose cohomology sheaves are coherent. We have an equivalence of categories*

$$D^b(X) \cong D^b_{\mathsf{coh}}(\mathsf{QCoh}(X)).$$

Phew! We're saved. The upshot: **when working in $D^b(X)$, we actually apply injective/projective/locally free resolutions of quasicoherent sheaves, not just coherent sheaves.** Computing derived functors (see §5.2) and such can be done using resolutions of quasicoherent sheaves, and not some really bizarre constructions (if they exist at all).

The category $D^b(X)$ is very interesting and important, and will barely be touched upon here for all of its relevance and usage. Instead, we refer the reader to [Huy06] for a wonderful exposition of the topic!

Finally, we'll cap off this (sub)section by quoting some fundamental results which illustrate how $D^b(X)$ reveals much of the information about the original scheme X. Note: some of these results may involve terminology that we have not defined. I don't want to delve further into them (and instead refer you to [Huy06]), only to give an idea to the reader why $D^b(X)$ should be useful to studying X itself.

Proposition 4.5.3. *Let X be a noetherian scheme and $D^b(X)$ be its bounded derived category of quasicoherent sheaves.*

(1) $D^b(X)$ is an indecomposable triangulated category iff X is connected.

(2) Let X be a smooth projective variety. The homological dimension of $\mathsf{Coh}(X)$ is equal to the dimension of X.

(3) Let C be a smooth projective curve. Then the previous statement implies that any object in $D^b(C)$ is isomorphic to the direct sum of its (shifted) cohomology sheaves (which are coherent!).

(4) Let X be a projective variety over a field. Let $\mathcal{L}$ be an ample line bundle on X. Then the powers $\{\mathcal{L}^{\otimes n} \mid n \in \mathbb{Z}\}$ form an ample sequence in $\mathsf{Coh}(X)$ (and thus they form a spanning class in $D^b(X)$).

(5) Let X and Y be smooth projective varieties over a field. If there is an exact equivalence $D^b(X) \simeq D^b(Y)$, then $\dim X = \dim Y$ and their canonical bundles ω_X and ω_Y have the same order (in their Picard groups). If the (anti)canonical bundle of X is ample, then X and Y are isomorphic (and thus the (anti)canonical bundle of Y is also ample).

(6) Building off of the previous result, if there is a fully faithful exact functor $F : D^b(X) \to D^b(Y)$ which admits left and right adjoints, then in fact F has a very concrete description as a Fourier-Mukai transform.

(7) If $\mathsf{Coh}(X) \simeq \mathsf{Coh}(Y)$ for smooth projective varieties X, Y, then X and Y are isomorphic.

(8) Let X be a smooth projective variety with an ample (anti)canonical bundle. Then $\mathrm{Aut}(D^b(X)) \simeq \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$, where the $\mathbb{Z}$ is the shift functor, $\mathrm{Aut}(X)$ are (the automorphisms on $D^b(X)$ induced by) automorphisms of X, and $\mathrm{Pic}(X)$ are tensoring by line bundles on X.

5 Derived functors

We encounter lots of different kinds of functors between categories. Let's list some important ones here from algebraic geometry.

Example 5.0.1. (1) Let X be a scheme. Then the global sections functor $\Gamma(X, -) : \mathsf{QCoh}(X) \to \mathsf{AbGrp}$ along with its restriction $\Gamma(X, -) : \mathsf{Coh}(X) \to \mathsf{AbGrp}$ are left exact functors.

(2) Let $f : X \to Y$ be a morphism of schemes. Then $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ is left exact. Its restriction $f_* : \mathsf{Coh}(X) \to \mathsf{Coh}(Y)$ is also left exact.

(3) Let X be a scheme and $\mathcal{F} \in \mathsf{QCoh}(X)$. Then $\mathrm{Hom}(\mathcal{F}, -) : \mathsf{QCoh}(X) \to \mathsf{AbGrp}$ is left exact.

(4) Let X be a scheme and $\mathcal{F} \in \mathsf{QCoh}(X)$. Then $\mathcal{H}om(\mathcal{F}, -) : \mathsf{QCoh}(X) \to \mathsf{QCoh}(X)$ is left exact.

(5) Let $\mathcal{R}$ be a sheaf of commutative rings on a scheme X, and let $\mathsf{Sh}_{\mathcal{R}}(X)$ denote the abelian category of $\mathcal{R}$-modules. For $\mathcal{F} \in \mathsf{Sh}_{\mathcal{R}}(X)$, then $\mathcal{F} \otimes_{\mathcal{R}} - : \mathsf{Sh}_{\mathcal{R}}(X) \to \mathsf{Sh}_{\mathcal{R}}(X)$ is right exact.

(6) Let $f : X \to Y$ be a morphism of schemes. Then $f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is right exact, as is its restriction $f^* : \mathsf{Coh}(Y) \to \mathsf{Coh}(X)$. (Recall that $f^* \dashv f_*$ form an adjoint pair.)

These functors will (often) lift to functors in the derived category. But when we work with derived categories, we need functors to be exact, so that they send exact triangles to exact triangles. In this section we'll construct the **derived functors** to form derived versions of the usual functors, and these derived functors will enjoy the property of being exact.

5.1 Injectives, projectives, and resolutions

Not just any functors can be made into derived functors: we usually need to assume that the original functor is either left or right exact. The reason is that taking resolutions by projective or injective objects only "covers up" one side, so if the functor is left exact, then the non-exactness on the right side is patched up by the injective resolution going off to infinity on the right.

Since this section is mostly just technical details, I'll give a tl;dr: **replace your complex with a resolution.** There, now feel free to skip to §5.2.

Definition 5.1.1. An abelian category $\mathcal{A}$ contains **enough injectives** (respectively **enough projectives**) if for any object $A \in \mathcal{A}$, there exists an injective map $A \to I$ to an injective object I (respectively, a surjective map $P \to A$ from a projective object P).

Notation 5.1.2. Assume $\mathcal{A}$ has enough injectives (respectively projectives). Denote by $\mathcal{I}$ (respectively $\mathcal{P}$) the full additive subcategory of $\mathcal{A}$ of injective objects (respectively projectives). The corresponding constructions $K^*(\mathcal{I})$ and $K^*(\mathcal{P})$ follow.

One immediate corollary is that if $\mathcal{A}$ has enough injectives (respectively enough projectives) then any object $A \in \mathcal{A}$ has an injective resolution $0 \to A \to I^0 \to \cdots$ (respectively a projective resolution

$\cdots \to P^{-2} \to P^{-1} \to P^0 \to A \to 0$); in particular, we have a quasi-isomorphism $A \to I^\bullet$ (respectively $P^\bullet \to A$). In fact **this extends to any complex**, in that if $\mathcal{A}$ has enough injectives (respectively enough projectives), then **we always have a quasi-isomorphism** $A^\bullet \to I^\bullet$ of injectives (respectively $P^\bullet \to A^\bullet$ of projectives) in $K^+(\mathcal{A})$ (and therefore an isomorphism in $D^+(\mathcal{A})$). These quasi-isomorphisms play nicely with the maps $K^*(\mathcal{A}) \to D^*(\mathcal{A})$. This can be summarized via the following main technical tool:

Proposition 5.1.3. *Suppose that $\mathcal{A}$ has enough injectives (respectively, enough projectives). Then the natural functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$ is an equivalence (respectively, $K^-(\mathcal{P}) \to D^-(\mathcal{A})$ is an equivalence).*

Example 5.1.4. A familiar example goes back to algebraic topology: when computing Tor groups, we need to replace one of the factors by a free resolution. This is because the tensor product is right exact, so we "cover up" the lack of left exactness by allowing a (possibly) infinite resolution to the left! Finally, free modules are projective, so we're actually taking a projective resolution.

5.2 Derived functors

When we have an arbitrary functor between between abelian categories, it's not always easy (actually, it is, but it's usually not very *useful*) to define the derived functors. Remember that we only care about derived functors because they're supposed to "exactify" our previously partially-exact functors. If they don't, well, they lose most of the properties that make them candidates for the derived versions of what we're used to, but maybe they're not so meaningless.

The tl;dr: **for a left (respectively, right) exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories, we can "patch up" the non-exactness on the other side by taking an injective (respectively, projective) resolution and applying F to these resolutions, resulting in an exact functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ (respectively, $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$).** Note that if we instead work with the **bounded derieved category** $D^b(\mathcal{A})$, then we don't ever need to worry about the $+$ or $-$.

Ok, there are some technical points swept under the rug (namely, we can get away with non-injective and non-projective resolutions using acyclic objects, and in fact often need to), but the real tl;dr is **replace your object with the relevant resolution and apply your functor to the resulting complex.** Now let's see many important examples! For simplicity, I'll always work in the bounded derived category $D^b(\mathcal{A})$. Furthermore, whenever I take an injective resolution I'll implicitly assume that the category has enough injectives, and whenever I take a projective resolution I'll implicitly assume that the category has enough projectives. If you only want to see examples in $D^b(X)$, feel free to skip to the next subsection.

Example 5.2.1. (1) Let $M \in \mathcal{A}$. Then $\mathrm{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \mathsf{AbGrp}$ is left exact, so we make the right derived functor $R\mathrm{Hom}(M, -) : D^b(\mathcal{A}) \to D^b(\mathsf{AbGrp})$. This works by taking an injective resolution $B^\bullet \xrightarrow{\text{qis}} I^\bullet$ and applying $\mathrm{Hom}_{\mathcal{A}}(M, -)$ to this resolution. Upon further reflection, the cohomology groups of this resulting complex are precisely the Ext functors $\mathrm{Ext}^i_{\mathcal{A}}(M, -)$. In fact, (assuming $\mathcal{A}$ has enough injectives,) **there are natural isomorphisms $\mathbf{Ext}^i_{\mathcal{A}}(A, B) \simeq \mathbf{Hom}_{D(\mathcal{A})}(A, B[i])$.**

(2) Similarly, Tor groups are just the cohomology groups of the left derived tensor product. Suppose we have $A, B \in \mathcal{A}$. Then the **derived tensor product** $A \otimes^L B$ is defined by taking a projective resolution of either A or B (recall that both $A \otimes -$ and $- \otimes B$ are both right exact) and taking the tensor with the other factor as complexes. Then $\mathrm{Tor}_i^{\mathcal{A}}(A, B)$ is defined to be the ith cohomology group of the resulting complex. In the case of R-modules, we usually take a free resolution.

(3) Extending the previous examples, we define the Hom functor in $D(\mathcal{A})$ as follows. For $A^\bullet$ a complex, then $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ is the complex

$$\mathrm{Hom}^i(A^\bullet, B^\bullet) := \bigoplus_k \mathrm{Hom}(A^k, B^{k+i}), \quad d(f) := d_B \circ f - (-1)^i d_A.$$

This is well-behaved when $B^\bullet$ is a complex of injectives, hence we may define $R\mathrm{Hom}^\bullet(A^\bullet, -) : D^+(\mathcal{A}) \to D(\mathsf{AbGrp})$ by taking an injective resolution of the input and applying $\mathrm{Hom}^\bullet(A^\bullet, -)$. We define $\mathrm{Ext}^i(A^\bullet, B^\bullet) := H^i(R\mathrm{Hom}^\bullet(A^\bullet, B^\bullet))$ and, just as before, obtain natural isomorphisms $\mathrm{Ext}^i(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i])$.

(3) Similarly, for a complex $B^\bullet$, we can define $R\mathrm{Hom}^\bullet(-, B^\bullet) : D^-(\mathcal{A})^{\mathrm{op}} \to D(\mathsf{Ab})$. This defines a bifunctor

$$D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\mathsf{AbGrp})$$

which is exact in both inputs. If $\mathcal{A}$ has enough injectives **and** projectives, then the two bifunctors $R\mathrm{Hom}^\bullet(-,-)$ (in each input) give the same bifunctor

$$R\mathrm{Hom}^\bullet(-,-) : D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathsf{AbGrp}).$$

(4) The group cohomology of a group G is defined to be the right derived functor of the functor $M \mapsto M^G$ of invariants for a G-module M. Group homology is the left derived functor of the functor $M \mapsto M_G$ of coinvariants.

(5) Similarly, Lie algebra cohomology is the right derived functor of the invariants functor $(-)^{\mathfrak{g}}$, and Lie algebra homology is the left derived functor of the coinvariants functor $(-)_{\mathfrak{g}}$.

(6) Once again, Hochschild cohomology is the right derived functor of the invariants functor $(-)^A$, and Hochschild homology is the left derived functor of the coinvariants functor $(-)/[A,-]$.

(7) Étale cohomology also arises as a derived functor.

5.3 Derived functors on $D^b(X)$

Let X be a noetherian scheme. Recall that $\mathsf{QCoh}(X)$ has enough injectives. Let's see some important examples of derived functors on $D^b(X)$.

Example 5.3.1. (1) The **global sections functor** $\Gamma : \mathsf{QCoh}(X) \to \mathsf{Vec}(k)$ is a left exact functor. Since $\mathsf{QCoh}(X)$ has enough injectives, we can define $R\Gamma : D^+(\mathsf{QCoh}) \to D^+(\mathsf{Vec}(k))$ by replacing $\mathcal{F}^\bullet$ by an injective resolution and then applying Γ term by term. In fact, the sheaf cohomology groups $H^i(X, \mathcal{F}^\bullet)$ (note the similar notation to the cohomology functors/objects...) are precisely defined to be the cohomology groups of the complex of vector spaces $R\Gamma(\mathcal{F}^\bullet)$. For $\mathcal{F}^\bullet = \mathcal{F}$ a sheaf these are just the ordinary sheaf cohomology groups; for a complex, they are sometimes called the hypercohomology groups.

Since every complex of vector spaces splits, in fact $R\Gamma(\mathcal{F}^\bullet) \simeq \bigoplus_{n\in\mathbb{Z}} H^n(X, \mathcal{F}^\bullet)[-n]$ in $D^+(\mathsf{Vec}(k))$.

(2) Let $f : X \to Y$ be a map of noetherian schemes. The **direct image functor** $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ is a left exact functor. So naturally, $Rf_* : D^+(\mathsf{QCoh}(X)) \to D^+(\mathsf{QCoh}(Y))$ works by taking a complex $\mathcal{F}^\bullet$, replacing it by an injective resolution, and applying f_* to this complex.

The **higher direct images** $R^i f_*(\mathcal{F}^\bullet)$ are the cohomology sheaves of $Rf_*(\mathcal{F}^\bullet)$ (specifically the ith one in this case). If $\mathcal{F}$ is just a quasicoherent sheaf then the higher direct images $R^i f_*\mathcal{F}$ are quasicoherent sheaves on Y.

If f is proper, then $R^i f_*$ sends a complex of coherent sheaves to a complex of coherent sheaves, thus defining an exact functor $Rf_* : D^b(X) \to D^b(Y)$.

(3) Since $\mathsf{QCoh}(X)$ has enough injectives, we can define **the Hom complexes** $R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$ on $D^b(X)$ by replacing $\mathcal{E}^\bullet$ by an injective resolution (i.e., a quasi-isomorphic complex of injective sheaves) and applying $\mathcal{H}om(\mathcal{F}^\bullet, -)$. Typically there are not enough projective objects in $\mathsf{Coh}(X)$. However, we can actually use locally free sheaves in place of projective objects when resolving $\mathcal{F}^\bullet$. In other words, we may compute $R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$ either by resolving $\mathcal{F}^\bullet$ with locally free sheaves (technical remark: we need X to be regular to ensure that this resolution is bounded), or by resolving $\mathcal{E}^\bullet$ by injective sheaves, and then applying the appropriate $\mathcal{H}om$.

(4) As before, we define the **Ext complexes** to be $\mathcal{E}xt^i(\mathcal{F}^\bullet, \mathcal{E}^\bullet) := R^i\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet[i])$.

(5) We define the **dual complex** $\mathcal{F}^{\bullet\vee}$ of a complex $\mathcal{F}^\bullet \in D^b(X)$ to be $\mathcal{F}^{\bullet\vee} := R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{O}_X)$.

(6) We can define the **derived tensor product** $\mathcal{E}^\bullet \otimes^L \mathcal{F}^\bullet$ by taking a locally free resolution of both complexes and applying the usual tensor product of complexes (i.e., taking the total complex of the double complex). (This is essentially due to the fact that tensoring with a locally free sheaf on the level of coherent/quasicoherent sheaves is exact.) Just as in the $R\mathcal{H}om$ case, locally free objects are sufficient stand-ins for the lack of projective objects!

We also define **the Tor complexes** to be $\mathcal{T}or_i(\mathcal{F}^\bullet, \mathcal{E}^\bullet) := H^{-i}(\mathcal{F}^\bullet \otimes_L \mathcal{E}^\bullet)$. Note that when the inputs are just ordinary coherent sheaves, this agrees with the usual definition of Tor sheaves!

(7) Let $f : X \to Y$. We define the **inverse image functor** Lf^* as follows. Recall that $f^* = \Big(\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} (-)\Big) \circ f^{-1}(-)$. The f^{-1} is exact, while the other factor is right exact, hence f^* is right exact. Then we define $Lf^* := \Big(\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} (-)\Big) \circ f^{-1}(-)$.

6 Bounded t-structures and hearts

In this section, we take a quick detour to discuss some important notions in triangulated categories. Very roughly speaking, we have a notion of "degrees" of a complex $M^\bullet \in D(\mathcal{A})$ which are exactly those integers n for which the nth cohomology objects of $M^\bullet$ are nonzero. This allows us to understand the "nonnegative" and "nonpositive" objects in $D(\mathcal{A})$, as well as speak of $\mathcal{A} \hookrightarrow D(\mathcal{A})$ as the complexes concentrated in degree 0. The goal of (bounded) t-structures and hearts is to axiomatize these notions. It turns out that there are many other t-structures obeying the same important properties, but give very different viewpoints on the category. One important example is the notion of a perverse sheaf, which arises as a result of another t-stucture on $D(\mathcal{A})$! Furthermore, we discuss the important notion of **filtration by cohomology**.

6.1 Motivation from $D(\mathcal{A})$

We saw that the cohomology functors carry lots of important information. Indeed, we have two subcategories $D_{\geq 0}(\mathcal{A})$ and $D_{\leq 0}(\mathcal{A})$ of $D(\mathcal{A})$, which consist of complexes with no cohomology in negative degrees and no cohomology in positive degrees. These are induced by **truncation functors**, which do exactly as their name suggests: they truncate the complex at a certain point.

Definition 6.1.1. Define the **truncation functors** $\tau_{\leq i}$ and $\tau \geq j$ as follows. For a complex $M^\bullet \in D(\mathcal{A})$, then

$$(\tau_{\leq i} M^\bullet)^n := \begin{cases} M^n & n < i, \\ \ker(M^i \to M^{i+1}) & n = i, \\ 0 & n > i. \end{cases}$$

$$(\tau_{\geq j} M^\bullet)^n := \begin{cases} M^n & n \geq j, \\ \operatorname{coker}(X^{j-2} \to X^{j-1}) & n = j-1, \\ 0 & n < j-1. \end{cases}$$

Concretely, they just **truncate** the complex:

$$\tau_{\leq 0}(\cdots \to X^{-2} \to X^{-1} \to X^0 \to 0 \to X^2 \to X^3 \to X^4 \to\to \cdots) = \cdots \to X^{-2} \to X^{-1} \to X^0 \to 0 \to 0 \to \cdots.$$

Note that in particular, if $H^n(M^\bullet) = 0$ for all $n > i$, then $\tau_i(M^\bullet) \cong M^\bullet$ in $D(\mathcal{A})$, thus **replacing a (possibly infinite) complex with bounded cohomology by a complex which is bounded in the same way**.

A bit more thought and we see that for any $M^\bullet \in D(\mathcal{A})$, we have the exact triangle

$$\tau_{\leq 0} M^\bullet \to M^\bullet \to \tau_{\geq 1} M^\bullet \to \tau_{\leq 0} M^\bullet[1].$$

This is essentially constructing what is called the **standard t-structure** on $D(\mathcal{A})$. The **heart** is

$D_{\leq 0}(\mathcal{A}) \cap D_{\geq 0}(\mathcal{A}) = \mathcal{A} \subset D(\mathcal{A})$, and in this special example $D(\text{heart})$ is precisely the original category $D(\mathcal{A})$. Note that **this is not true in general!**

The notion of a t**-structure** is a generalization of these crucial properties.

6.2 t-structures

Let $\mathcal{D}$ be a triangulated category.

Definition 6.2.1. Let $\mathcal{D}_{\leq 0}$ and $\mathcal{D}_{\geq 0}$ be full subcategories of $\mathcal{D}$ closed under isomorphism (i.e. saturated). Define $\mathcal{D}_{\leq n} := \mathcal{D}_{\leq 0}[-n]$ and $\mathcal{D}_{\geq n} := \mathcal{D}_{\geq 0}[-n]$. Then $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ defines a t**-structure on** $\mathcal{D}$ if the following conditions are satisfied:

(1) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0}) = 0$.
(2) $\mathcal{D}_{\leq 0} \subseteq \mathcal{D}_{\leq 1}$ and $\mathcal{D}_{\geq 1} \subseteq \mathcal{D}_{\geq 0}$. Explicitly, for any object $X \in \mathcal{D}_{\leq 0}$, then $X[1]$ is also in $\mathcal{D}_{\leq 0}$ (and similarly for $\mathcal{D}_{\geq 0}$). This is equivalent to $\mathcal{D}_{\leq n} \subseteq \mathcal{D}_{\leq m}$ for any $n < m$, and similarly for $\geq$.
(3) For every object $X \in \mathcal{D}$, we have an exact triangle $A \to X \to B \to A[1]$ where $A \in \mathcal{D}_{\leq 0}$ and $B \in \mathcal{D}_{\geq 1}$, which splits X into the "nonpositive" and "positive" parts.

As mentioned before, this definition is deeply inspired by the example from $D(\mathcal{A})$!

Example 6.2.2. Let $\mathcal{D} = D(\mathcal{A})$ be the derived category of some abelian category $\mathcal{A}$. The **standard** t**-structure** is constructed as follows. The subcategories $\mathcal{D}_{\leq n}$ are just complexes supported in degrees at most n, and $\mathcal{D}_{\geq n}$ are complexes supported in degrees at least n. The second condition is clear: the functor $X \mapsto X[1]$ just shifts the entire complex one degree down, which clearly preserves the property of being supported only in nonpositive degree. The first and third conditions follow from truncation functors.

Definition 6.2.3. A t-structure is **bounded** if

$$\bigcap_{n \in \mathbb{Z}} \mathcal{D}_{\leq n} = \bigcap_{n \in \mathbb{Z}} \mathcal{D}_{\geq n} = \{0\}.$$

Example 6.2.4. If we restrict the standard t-structure to the bounded derived category (i.e. the full subcategory of bounded complexes) $D^b(\mathcal{A}) \subset D(\mathcal{A})$, then we get the standard t-structure on $D^b(\mathcal{A})$. By definition no bounded complex has infinite cohomology in either direction, so the standard t-structure on $D^b(\mathcal{A})$ is a bounded t-structure. (It is not on $D(\mathcal{A})$!)

In fact, we always have an analogue of the truncation functors from $D(\mathcal{A})$ in any triangulated category carrying a t-structure, and these truncation functors essentially explain the structure of the $\mathcal{D}_{\leq n}$ and $\mathcal{D}_{\geq n}$.

Proposition 6.2.5. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then there are truncation functors $\tau_{\leq i}$ and $\tau_{\geq j}$ satisfying $\tau_{\leq i} X \in \mathcal{D}_{\leq i}$ and $\tau_{\geq j} X \in \mathcal{D}_{\geq j}$ for any $X \in \mathcal{D}$. They satisfy the property that for* *any $X \in \mathcal{D}$ and any N, we have exact triangles*

$$\tau_{\le N}X \to X \to \tau_{\ge N+1}X \to \tau_{\le N}X[1].$$

In fact, they are precisely the adjoints of the inclusions of categories: $\tau_{\le N}$ is the right adjoint to the inclusion $\mathcal{D}_{\le N} \hookrightarrow \mathcal{D}$, and $\tau_{\ge N}$ is the left adjoint to the inclusion $\mathcal{D}_{\ge N} \hookrightarrow \mathcal{D}$.

Feel free to just skip to the next subsection if you don't care about truncation functors and want to know what a heart is. But if you're interested in how the truncation functors work, the rest of this subsection will explain that.

Proof. We can prove it as follows. First we construct a candidate for each $\tau_{\le N}X$ and $\tau_{\ge N}X$, which are given by condition (3) in Definition 6.2.1 (by translating). We then show the functoriality of the τ, i.e. that they are adjoints to the inclusion of categories. Let me illustrate it in the case of $\mathcal{D}_{\ge N}$, since it is exactly the same for $\tau_{\le N}$. We have the exact triangle

$$\tau_{\le N-1}X \to X \to \tau_{\ge N}X \to \tau_{\le N-1}X[1],$$

but we don't know (yet) that these objects are unique. Now we obtain a long exact sequence using $\mathrm{Hom}_{\mathcal{D}}(-, Y)$ for $Y \in \mathcal{D}_{\ge N}$:

$$0 = \mathrm{Hom}_{\mathcal{D}}(\tau_{\le N-1}X[1], Y) \to \mathrm{Hom}_{\mathcal{D}}(\tau_{\ge N}X, Y) \to \mathrm{Hom}_{\mathcal{D}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\tau_{\le N-1}X, Y) = 0.$$

The first and last terms vanish by the conditions in Definition 6.2.1. Thus we get a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\tau_{\ge N}X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(X, Y).$$

The remaining step is to note that both $\tau_{\ge N}X$ and Y live in $\mathcal{D}_{\ge N}$, which is a full subcategory of $\mathcal{D}$, hence we obtain

$$\mathrm{Hom}_{\mathcal{D}_{\ge N}}(\tau_{\ge N}X, Y) = \mathrm{Hom}_{\mathcal{D}}(\tau_{\ge N}X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(X, Y),$$

which shows that $\tau_{\ge N}$ indeed is the left adjoint to the inclusion $\mathcal{D}_{\ge N} \hookrightarrow \mathcal{D}$. □

Example 6.2.6. Let $M^\bullet \in D(\mathcal{A})$ be a complex, where $D(\mathcal{A})$ is equipped with the standard t-structure. Then $\tau_{\le N}M^\bullet$ is a complex whose cohomology objects H^i are the same as those of $H^i(M^\bullet)$ for $i \le N$, and zero for $i > N$. The same result holds for $\tau_{\ge N}M^\bullet$.

Corollary 6.2.6.1. *$X \in \mathcal{D}_{\le 0} \iff \tau_{\ge 1}X = 0$. In other words, $\mathcal{D}_{\le 0}$ and $\mathcal{D}_{\ge 1}$ are orthogonal complements to each other.*

Of course, this is just a really abstract way of saying "an integer is nonnegative iff it is not positive!" Mathematicians sure like to do things the hard way.

Corollary 6.2.6.2. *$\mathcal{D}_{\ge N}$ and $\mathcal{D}_{\le N}$ are closed under extensions.*

Let me list two more properties of the truncation functors.

Proposition 6.2.7. *1. For $a \leq b$, then $\tau_{\leq a} \circ \tau_{\leq b} = \tau_{\leq a}$.*

2. For $a \leq b$, $\tau_{\geq a} \circ \tau_{\leq b} = \tau_{\leq b} \circ \tau_{\geq a}$.

The first is easily seen using adjoints. The second is from the octahedral axiom TR4.

One very important glimpse of future sections is by looking at Proposition 6.2.7 through the lens of $D(\mathcal{A})$.

Example 6.2.8. In Example 6.2.6, we saw that the truncation functors "split" complexes in $D(\mathcal{A})$ into a "lower" part and an "upper" part, with respect to cohomology. Upon closer reflection, we in fact see that $H^N(M^\bullet)$ is almost $\tau_{\leq N}\tau_{\geq N}M^\bullet$ – the former is an object of $\mathcal{A}$, which is identified with complexes in $D(\mathcal{A})$ concentrated in degree 0, while the latter is an object which is concentrated in degree N. The "objects" themselves match, up to shifting! This is summarized in the following proposition. Although an arbitrary triangulated category doesn't have cohomology functors, **a t-structure allows us to view cohomology as just truncating around N**. We'll see this laid out in Definition 6.4.4.

Corollary 6.2.8.1. *On $D(\mathcal{A})$, the cohomology functors H^i are just $\tau_{\leq i}\tau_{\geq i}(-)[-i]$, after taking the indentification $\mathcal{A} \leftrightarrow \{$complexes in $D(\mathcal{A})$ concentrated in degree zero$\}$. An analogous statement will be true for an arbitrary triangulated category $\mathcal{D}$ with some t-structure: see Definition 6.4.4.*

Finally, one last remark that makes the truncation functors more concrete: we have

$$\tau_{\leq N}X = (\tau_{\leq 0}(X[N]))[-N],$$

and analogously for $\tau_{\geq N}$. In other words, **just as in the case of complexes, to cut off at N we just shift downwards by N and cut off at 0, then shift back by N.**

6.3 Filtration by cohomology

Let's return to the setting of $D(\mathcal{A})$, the derived category of an abelian category $\mathcal{A}$. Our original motivation for using complexes, rather than simply the cohomology objects, is that **complexes remember more information than all of the cohomology objects**. Let's start with an important example.

Example 6.3.1. Let R be a ring. One useful way to study an R-module M is by considering its composition series. We take a filtration of M by submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M$ such that each M_{i+1}/M_i is a simple R-module. The **Jordan-Hölder theorem** implies that the set of the isomorphism classes of the quotients, counted with multiplicity, is uniquely determined by M and *independent of the filtration used.*

Of course, the catch is that the composition series, while simple to describe, carries less information than the original module itself. Indeed, we have very simple examples taken from Ext^1 already: if $\mathrm{Ext}^1_R(M_1, M_2) \neq 0$ for two (not necessarily distinct) simple modules M_1, M_2, then the modules whose simple quotients in the composition series is $\{M_1, M_2\}$ is in bijection with $\mathrm{Ext}^1_R(M_1, M_2)$. For a particularly simple example, the $\mathbb{Z}$-modules $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are already nonisomorphic, yet have the same simple quotients in their composition series, namely $\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$.

One useful way to think about complexes is that **complexes are analogous to the original module, while the cohomology objects are analogous to the composition series**. It stands to reason that we should be able to "filter" our complex by the cohomology objects, akin to the filtration in the composition series. This is precisely the case, and very reasonably, the tool used to isolate the cohomology objects one at a time is the truncation functor from Definition 6.1.1.

Let us now make this concrete. Let $\mathcal{A}$ be an abelian category and $D^b(\mathcal{A})$ be the bounded derived category. Let $M^\bullet \in D^b(\mathcal{A})$, and suppose that the cohomology groups $H^i(M^\bullet) \neq 0 \iff i \in I$ for some finite set I. Then let $n = \max(I)$. Then Proposition 6.2.5 tells us that we have the exact triangle

$$\tau_{\leq n-1} M^\bullet \to M^\bullet \to \tau_{\geq n} M^\bullet \to \tau_{\leq n-1} M^\bullet[1].$$

As explained in Example 6.2.8 and Corollary 6.2.8.1, we have that $H^n(M^\bullet) = \tau_{\leq n}\tau_{\geq n} M^\bullet$. But since the cohomology $H^i(\tau_{\geq n} M^\bullet)$ vanishes for $i > n$, it follows that $\tau_{\leq n}\tau_{\geq n} M^\bullet \cong \tau_{\geq n}$. Thus the exact triangle becomes

$$\tau_{\leq n-1} M^\bullet \to M^\bullet \to H^n(M^\bullet)[-n] \to \tau_{\leq n-1} M^\bullet[1].$$

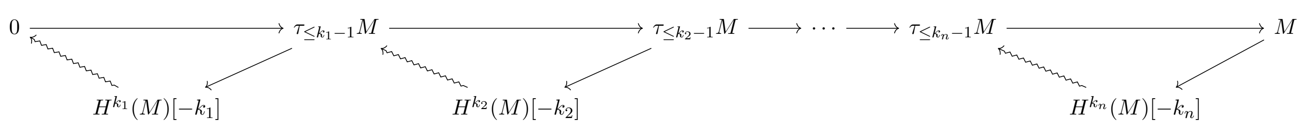
Now we iterate this process. The object $\tau_{\leq n-1} M^\bullet =: N^\bullet$ is again a bounded complex whose cohomology is nonzero precisely for the set $I - \{n\}$; its maximum nonzero cohomology is strictly smaller than $n := \max(I)$. This process must terminate since **at each step, we strip off the highest cohomology object of the complex**, and by assumption the cohomology is nonzero only for finitely many indices. We eventually obtain an exact triangle of the form

$$\tau_{\leq m-1} C^\bullet \to C^\bullet \to H^m(C^\bullet)[-m] \to (\tau_{\leq m-1} C^\bullet)[1],$$

where $C^\bullet$ is (quasi-isomorphic to) a complex concentrated in degree m. Then $\tau_{\leq m-1} C^\bullet = 0 \in D^b(\mathcal{A})$, so this process has terminated.

In conclusion, we find that

Proposition 6.3.2. *Let $M \in D^b(\mathcal{A})$ be a complex. Let $k_1 < k_2 < \cdots < k_n$ be the indices where M has nonzero cohomology. Then there is a sequence of maps*



where each triangle $\tau_{\leq k_{i-1} - 1} M \to \tau_{\leq k_i - 1} M \to H^{k_i}(M)[-k_i] \to \tau_{\leq k_{i-1}-1} M$ is an exact triangle.

This is known as **the filtration of M by its cohomology objects**. It bears significant resemblance to the notion of building some object from its composition series as a possibly non-trivial extension!

Remark 6.3.3. It turns out that this notion is quite useful: many spectral sequence arguments can actually be proven using the filtration by cohomology.

6.4 Heart of a t-structure

Definition 6.4.1. Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ be a t-structure. Then the **heart** of this t-structure, denoted by $\mathcal{D}_{\heartsuit}$, is defined to be $\mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$.

Example 6.4.2. Letting $\mathcal{D} = D(\mathcal{A})$ with the standard t-structure, we have $D(\mathcal{A})_{\heartsuit} = \mathcal{A} \subset D(\mathcal{A})$ under the canonical identification.

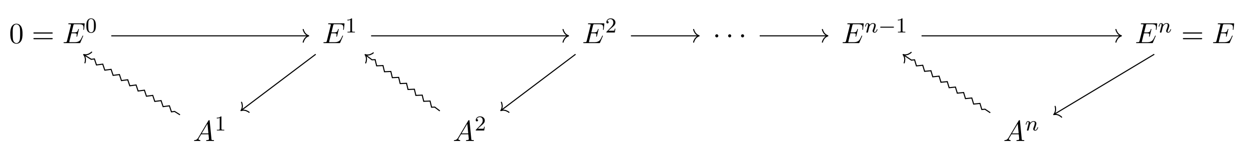
A major result is that **the heart is always an abelian category**. (The interested reader may consult Akhil Mathew's blog post about BDD to see how to construct the kernel and cokernel.) Unfortunately, $D(\mathcal{D}_{\heartsuit})$ **usually does not recover** $\mathcal{D}$**!** In fact, in many cases one cannot even define a map between these two categories.

In the case when the t-structure is bounded, there is an equivalent definition of the heart:

Definition 6.4.3 (Heart of a bounded t-structure)**.** Let $\mathcal{D}$ be a triangulated category with a bounded t-structure given by $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$. Then the **heart** of this bounded t-structure is a full abelian subcategory $\mathcal{A}^{\#} \subset \mathcal{D}$ such that

(1) For $k_1 > k_2$, then $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}^{\#}[k_1], \mathcal{A}^{\#}[k_2]) = 0$.

(2) For every nonzero $E \in \mathcal{D}$, there exist integers $k_1 > k_2 > \cdots > k_n$ and a sequence of exact triangles



such that each triangle in the diagram $E^i \to E^{i+1} \to A^i \to E^i[1]$ is an exact triangle (the $\rightsquigarrow$ arrows represent the connected morphism $C \to A[1]$ in exact triangles $A \to B \to C \to A[1]$), and $A^i \in \mathcal{A}^{\#}[k_1]$.

Notice that this definition bears *serious* resemblance to the filtration of a bounded complex by its cohomology objects, discussed in §6.3 and Proposition 6.3.2. Indeed, this lends itself to the following:

Definition 6.4.4. (t-cohomology) Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$. The **zeroth** t**-cohomology** of an object $M \in \mathcal{D}$ is defined to be ${}^tH^0(M) \coloneqq \tau_{\leq 0}\tau_{\geq 0} M \in \mathcal{D}_{\heartsuit}$. For any $n \in \mathbb{Z}$, the n**th** t**-cohomology** is defined to be the functor ${}^tH^n(-) : M \mapsto {}^tH^0(M[n]) = (\tau_{\leq n}\tau_{\geq n} M)[n] \in \mathcal{D}_{\heartsuit}$.

These are called the **cohomology objects of** M and if the t-structure is a bounded t-structure, **these cohomology objects are exactly the** A^i **appearing in Definition 6.4.3.** These t-cohomology functors really let us reconstruct everything we know and love about the derived category (namely, working very concretely with complexes and their indices); **a triangulated category with a t-structure behaves very similarly to the standard t-structure on a derived category!** Let's see how:

Proposition 6.4.5. *(1) A bounded t-structure is uniquely determined by its heart. (On the other hand, $D(\mathcal{A})$ is uniquely determined by $\mathcal{A}$.)*

(2) ${}^tH^0$ takes exact triangles in $\mathcal{D}$ to long exact sequences in $\mathcal{D}_\heartsuit$. (This is also true for $D(\mathcal{A})$, with the functor H^0.)

(3) A map f in $\mathcal{D}$ is an isomorphism iff all of the induced maps on ${}^tH^i$ are isomorphisms for each $i \in \mathbb{Z}$. (This is just the definition of a quasi-isomorphism for complexes.)

(4)

$$\mathcal{D}_{\leq N} = \{M \in \mathcal{D} \mid {}^tH^n(M) = 0 \text{ for all } n > N\},$$
$$\mathcal{D}_{\geq N} = \{M \in \mathcal{D} \mid {}^tH^n(M) = 0 \text{ for all } n < N\}.$$

(We have the exact same statement in the derived category.)

Therefore, we can imagine that a triangulated category $\mathcal{D}$ (equipped with a bounded t-structure) as being "almost" a direct sum $\bigoplus_{n\in\mathbb{Z}} \mathcal{D}_\heartsuit[n]$, in the sense that it is put together through finitely many parts, each living in some $\mathcal{D}_\heartsuit[n]$. (Of course, the way that they are put together is very complicated and akin to Ext groups!) Therefore we might imagine the following picture:

$$\cdots \boxed{\mathcal{A}[4]}\boxed{\mathcal{A}[3]}\boxed{\mathcal{A}[2]}\boxed{\mathcal{A}[1]}\boxed{\mathcal{A}[0]}\boxed{\mathcal{A}[-1]}\boxed{\mathcal{A}[-2]}\boxed{\mathcal{A}[-3]}\boxed{\mathcal{A}[-4]} \cdots .$$

Different t-structures give us different ways to view and understand our category. For example, **perverse sheaves** arise from yet another t-structure on $D^b(X)$, and they turn out to be very useful and important indeed!

6.5 Torsion pairs

Hopefully we agree that t-structures are very interesting and potentially useful in understanding triangulated categories. But how exactly do we find t-structures? Do we wait for geniuses to come up with new ideas?

Fortunately, there is a method to generate many nontrivial t-structures. It is known as **tilting at a torsion pair.**

Definition 6.5.1. Let $\mathcal{A}$ be an abelian category. A pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories is a **torsion pair** if

(i) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$, and

(ii) Every $E \in \mathcal{A}$ fits into a short exact sequence $0 \to T \to E \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. (Note that property (1) implies that this short exact sequence is automatically unique and functorial in E.)

In accordance with the picture in the previous subsection, we might have the following updated picture, bringing the torsion pair into account:

$$\cdots \underbrace{\boxed{\mathcal{T}[2]}\boxed{\mathcal{F}[2]}}_{\mathcal{A}[2]}\underbrace{\boxed{\mathcal{T}[1]}\boxed{\mathcal{F}[1]}}_{\mathcal{A}[1]}\underbrace{\boxed{\mathcal{T}[0]}\boxed{\mathcal{F}[0]}}_{\mathcal{A}[0]}\underbrace{\boxed{\mathcal{T}[-1]}\boxed{\mathcal{F}[-1]}}_{\mathcal{A}[-1]}\underbrace{\boxed{\mathcal{T}[-2]}\boxed{\mathcal{F}[-2]}}_{\mathcal{A}[-2]}\cdots.$$

The **canonical example** (from which it derives its name) $\mathcal{A} = \mathsf{Coh}(X)$, with $\mathcal{T}$ being torsion sheaves and $\mathcal{F}$ be torsion-free sheaves. There are many other examples in quivers, but let us skip those and discuss a more involved example, which can be safely skipped if you only care about derived categories (and not at all about stability conditions). I am covering it because it will come up in our seminar on Bridgeland stability, following [MS17], but it is not as relevant to the general discussion on derived categories.

Example 6.5.2. Let C be a smooth projective curve and let $\mathcal{A} = \mathsf{Coh}(C)$. For a vector bundle $\mathcal{E}$ on C, define its **slope** to be $\mu(\mathcal{E}) \coloneqq \frac{\deg(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}$. For $\lambda \in \mathbb{Q}$, we say that $\mathcal{E}$ **is semistable of slope** λ if $\mu(\mathcal{E}) = \lambda$ and every nonzero subbundle $\widetilde{\mathcal{E}} \subset \mathcal{E}$ has slope $\leq \lambda$. It turns out that every $\mathcal{E}$ has a unique **Harder-Narasimhan filtration** (HN filtration), which is a filtration

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subsetneq \cdots \subseteq \mathcal{E}_m = \mathcal{E}$$

such that each $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable of slope λ_i and the λ_i are strictly decreasing, i.e. $\lambda_1 > \lambda_2 > \cdots > \lambda_m$.

Let $\mu \in \mathbb{R}$. Then $\mathcal{A}_{\geq\mu}$ be the full subcategory consisting of vector bundles whose HN filtration quotients all have slope $\geq \mu$, and similarly let $\mathcal{A}_{<\mu}$ be the full subcategory consisting of vector bundles whose HN filtration quotients all have slope $< \mu$. It turns out that $(\mathcal{A}_{\geq\mu}, \mathcal{A}_{<\mu})$ is a torsion pair. The Hom-vanishing is due to the fact that if all of the slopes (of the quotients in the HN filtration) of $\mathcal{E}$ are strictly less than all of the slopes (of the quotients in the HN filtration) of $\mathcal{F}$, then $\operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) = 0$. Cool!

For more information on the Harder-Narasimhan filtration, see Jacob Lurie's notes.

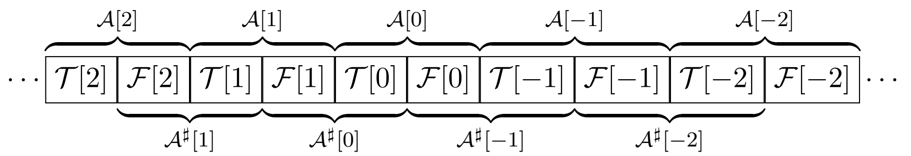
The interesting point is that **torsion pairs create new hearts from old hearts**.

Proposition 6.5.3. *Let $\mathcal{D}$ be a triangulated category with bounded t-structure. Let $\mathcal{A} = \mathcal{D}_\heartsuit$ be its heart. Suppose we have a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$. Then*

$$\mathcal{A}^\sharp \coloneqq \{E \in \mathcal{D} \mid {}^tH^0(E) \in \mathcal{T}, {}^tH^{-1}(E) \in \mathcal{F}, {}^tH^i(E) = 0 \quad \textit{for } i \neq 0, -1\}$$

defines a heart of a (different) bounded t-structure in $\mathcal{D}$. (Replace $\mathcal{D}$ by $D(\mathcal{A})$ and ${}^tH^i$ by H^i for the concrete example of a derived category.)

While objects in $\mathcal{A}$ are extensions of some $F \in \mathcal{F}$ by some $T \in \mathcal{T}$, hence determined by an element in $\operatorname{Ext}^1(F, T)$, objects in $\mathcal{A}^\sharp$ are extensions of some T by some $F[1]$, hence determined by some element in $\operatorname{Ext}^1(T, F[1]) = \operatorname{Ext}^2(T, F)$. It is worth keeping the following picture in mind.

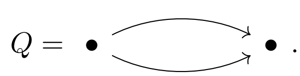


There are no morphisms going from left to right, and any object can be written as successive extensions of objects in the building blocks, starting from the right and extending to the left. Now the name "tilting" becomes clear!

We can repeatedly tilt a bounded t-structure to iteratively obtain new bounded t-structures. In the case of $\mathcal{D} = D^b(\mathcal{A})$, we have a canonical starting point of the standard t-structure. In fact, this process does result in quite a few t-structures. Arend Bayer summarizes the power of tilting to generate new t-structures in this MathOverflow post. In particular, if $\mathcal{A}, \mathcal{A}^\sharp$ are hearts of bounded t-structures and $\mathcal{A}^\sharp$ is contained in the extension closure $\langle \mathcal{A}, \mathcal{A}[1] \rangle$, then in fact we can obtain $\mathcal{A}^\sharp$ by a tilt of $\mathcal{A}$: explicitly, it's the tilt of the torsion pair $(\mathcal{A} \cap \mathcal{A}^\sharp, \mathcal{A} \cap \mathcal{A}^\sharp[-1])$.

It turns out that in algebraic geometry, **tilting is intricately related to quivers**. Let's see a very interesing example.

Example 6.5.4. Consider the abelian category $\mathsf{Coh}(\mathbb{P}^1)$. Let $(\mathcal{T}, \mathcal{F}) = (\mathcal{A}_{\geq 0}, \mathcal{A}_{<0})$ (see Example 6.5.2); we obtain the tilted heart $\mathcal{A}^\sharp$. Also consider the quiver



The abelian categories $\mathsf{Coh}(\mathbb{P}^1)$ and $\mathsf{Rep}(Q)$ are very different. For example, $\mathsf{Rep}(Q)$ has Jordan-Hölder filtrations, while $\mathsf{Coh}(\mathbb{P}^1)$ does not. However, *their derived categories are equivalent*:

$$\Phi_T : D^b(\mathsf{Coh}(\mathbb{P}^1)) \xrightarrow{\sim} D^b(\mathsf{Rep}(Q)).$$

This equivalence is defined by the **tilting sheaf** $T = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, which will be discussed shortly. Furthermore, $\Phi_T(\mathcal{A}^\sharp) = \mathsf{Rep}(Q) \subset D^b(\mathsf{Rep}(Q))$.

So using different t-structures, two very different abelian categories can both exist as hearts of the same triangulated category!

Definition 6.5.5 (Tilting sheaf)**.** Let X be a smooth projective $\mathbb{C}$-variety. Then a coherent sheaf T on X is a **tilting sheaf** if:

(i) The **tilting algebra** $A = \operatorname{End}_{\mathcal{O}_X}(T)$ has finite global dimension.

(ii) $\operatorname{Ext}^k_{\mathcal{O}_X}(T, T) = 0$ for all $k > 0$.

(iii) Using "classical operations" (i.e., cones, direct summands, shifts, etc.) we can generate all of $D^b(\mathsf{Coh}(X))$ from T.

Theorem 6.5.6 (Baer, Bondal)**.** *Let T be a tilting sheaf on X and let $A = \operatorname{End}_{\mathcal{O}_X}(T)$ be the tilting algebra. Then the categories $D^b(\mathsf{Coh}(X))$ and $D^b(A\text{–}\mathsf{mod})$ are equivalent.*

More specifically, they are induced from functors $F \coloneqq \mathrm{Hom}_{\mathcal{O}_X}(T, -)$ *and* $G \coloneqq - \otimes_A T$*:*

$$RF : D^b(\mathsf{Coh}(X)) \xrightarrow{\sim} D^b(A\text{−}\mathsf{mod}),$$
$$Rg : D^b(A\text{−}\mathsf{mod}) \xrightarrow{\sim} D^b(\mathsf{Coh}(X)).$$

In Example 6.5.4, $A = \mathrm{End}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{C}Q$, where $\mathbb{C}Q$ is the path algebra of Q. There are many more complicated examples; for example, see [Cra07].

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